Worksheet 2:

a) **The frequency of the gravitational waves**

Some of the orbital parameters of the two black holes are closely related to the frequency of the gravitational waves detected by the interferometers. The frequency of the gravitational waves is exactly twice the orbital frequency of the two massive celestial bodies. This key result, $f_{GW} = 2 \cdot f_{Orb}$, can easily be demonstrated by a simple computer simulation. Start up the program *Gravitationswellen.exe*[1] and compare the motion of the two masses to the motion of the emitted waves. As you can see, a new wave peak forms every half cycle.

Remark: It is important to bear in mind that using a two-dimensional grid to represent the curvature of space and the propagation of gravitational waves is nothing more than a simplification that attempts to make the physical relationships easier to understand. In reality, we would need to work with spherical waves that influence all three spatial dimensions, as well as time. There is no simple way to fully represent general-relativistic space-time on a graph!

b) **The distance between the black holes**

Before they merged together, the two circling black holes had masses equivalent to 29 times and 36 times the mass of the Sun ($M_{Sun} = 2 \cdot 10^{30}$ kg). To simplify the calculations below, assume that the two black holes have equal mass and that the total mass is equal to 65 times the solar mass. In other words, $M_{total} = 65 \cdot M_{Sun}$.

With the assumption that the masses are equal, the shared centre of gravity is located exactly half-way between them. Thus, the radius of the orbit of each mass is $d/2$, but the distance between them is d in the gravitational equations.

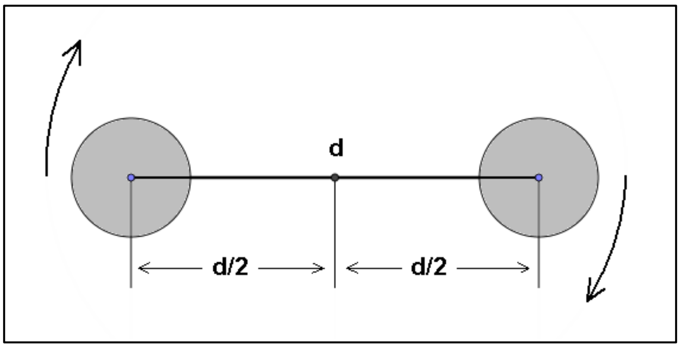

Image source: M. Borchardt

[1] http://www.mabo-physik.de/gravitationswellen.html

1. Prove the following formula for the distance between the two masses

$$d = \left(\frac{G \cdot M_{total}}{\pi^2 \cdot f_{GW}^2} \right)^{\frac{1}{3}}.$$

You can use the formulae listed below:

- the law of gravitation, $F_G = \frac{G \cdot m_1 \cdot m_2}{r^2}$, where $m_1 = m_2 = \frac{M_{total}}{2}$ and $r = d$;
- the centripetal force, $F_{CP} = \frac{m \cdot v^2}{r}$, where $m = \frac{M_{total}}{2}$ and $r = \frac{d}{2}$;
- the orbital velocity about the common centre of gravity, $v = 2\pi \cdot r \cdot f_{Orb}$, where $r = \frac{d}{2}$ and $f_{Orb} = \frac{f_{GW}}{2}$.

Hint:

Observe that the gravitational force acts as a centripetal force, so $F_{CP} = F_G$, then replace the velocity v with the formula $v = 2\pi \cdot r \cdot f_{Orb}$ at a suitable point in your argument.

2. The two graphs on the right show the gravitational wave recordings of the interferometer at Hanford, as well as the theoretical values predicted by general relativity.

   Calculate the approximate value of the distance between the two masses in the interval from 0.35 to 0.4 seconds. You can do this by estimating the frequency of the gravitational wave on this interval, then using the formula from b) 1. to find the distance d in km.

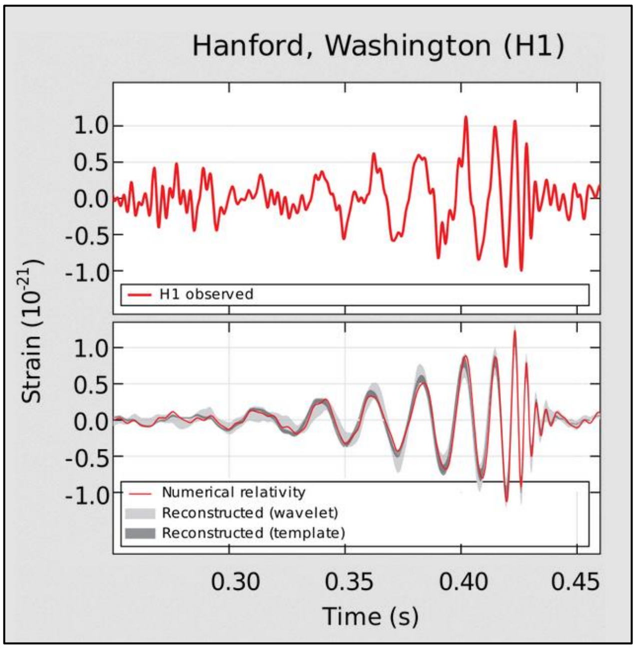

Image source: https://commons.wikimedia.org/wiki/File:LIGO_measurement_of_gravitational_waves.svg

3. The distance is usually expressed in terms of the Schwarzschild radius R_S, which can be calculated from the formula $R_S = \frac{2 \cdot G \cdot M_{total}}{c^2}$.

   Express the Schwarzschild radius in km, then given the distance d between the two black holes in units of R_S.

4. The diagram on the right plots the distance ("separation") between the two black holes over time. This was calculated from the frequency of the gravitational wave by applying the equations of general relativity.

   Check whether the value you calculated earlier is consistent with the theoretical values during the relevant time interval.

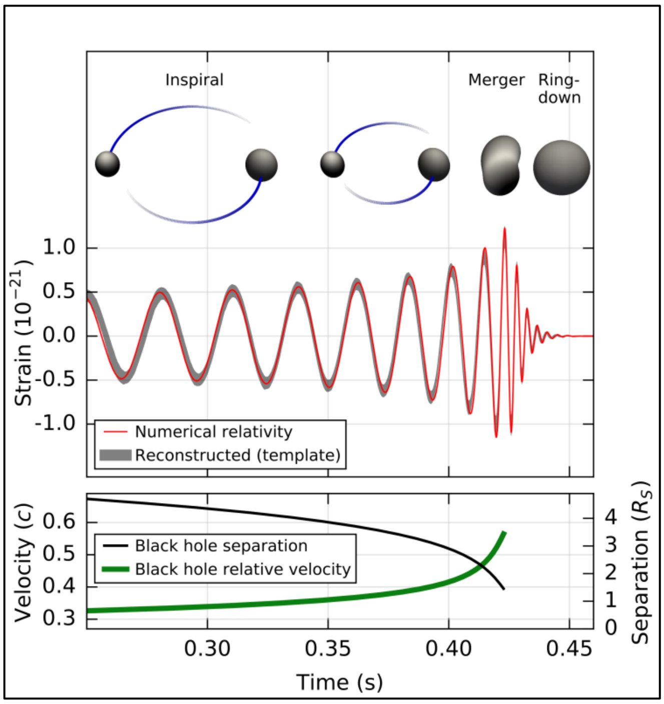

Image source: B. P. Abbott et al. (2016). "Observation of Gravitational Waves from a Binary Black Hole Merger". In: *Physical Review Letters* 116:06. https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.116.061102

c) **The orbital speed of the black holes**

To find the orbital speed of the two masses about their shared centre of gravity, you can substitute the distance d (in metres) calculated earlier into the following formula:

$$v_{Orb} = \frac{1}{2} \cdot \sqrt{\frac{G \cdot M_{total}}{d}} .$$

Start by proving this formula, then calculate the orbital speed. Express your answer as a percentage of the speed of light.

Remark:
If you compare your result with the speed curve in the figure above, you will notice that your value is significantly smaller than it should be. The correct value for the speed is exactly twice the value calculated from the above formula for v_{Orb}.

In fact, this shows that the laws of classical physics – more specifically, Newton's law of gravitation in this case – are violated, and the theory of general relativity[2] is needed to explain the experimental results. General relativity predicts that an additional factor of two is needed under certain extreme gravitational conditions, such as when two black holes are merging together.

[2] In practice, research into gravitational waves usually uses a "Post-Newtonian approximation" (PN) that is much more suitable for numerically implementing Einstein's field equations on supercomputers.

